J. Fluid Mech. (1982), *vol.* 116, *pp.* 459–473
Printed in Great Britain

Elastic jumps on fluid-filled elastic tubes

By S. J. COWLEY†
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

(Received 4 August 1980 and in revised form 27 August 1981)

This paper is concerned with fluid flows through membranous elastic tubes. The tubes are assumed to be either untethered (except at the ends), or to be tethered by axial forces that prevent all axial motion of the tube.

First we verify, for axisymmetric deformations that vary slowly in the axial direction, that the elastic balance of the tube can be expressed in terms of a 'tube law'. In the case of the tethered tubes, the tube law takes the widely used form of a pressure/area relation for both steady and unsteady deformations. However, for untethered tubes, the tube law will generally be time-dependent if the deformations are unsteady. Conditions are then derived between the up- and downstream flows of a turbulent elastic jump. Although it is necessary for the elastic balance to be described by a tube law far up- and far downstream of the jump, we do not assume that the tube law is valid inside the jump. The conditions we derive are believed to hold for both collapsed and expanded tubes.

The theory has applications in describing fluid flows within, for example, the airways, blood vessels and the urethra.

1. Introduction

In recent years there have been many studies of fluid flow through flexible tubes, motivated, for example, by the physiological problems of blood flow through arteries and veins, and air flow from the lungs. An extensive list of applications has been given by Shapiro (1977*b*), with particular reference to collapsed tubes.

An often-used model of such flows is based on the assumptions that the fluid is incompressible and inviscid, that the flow is irrotational, and that all disturbances have a long wavelength. Further, if the tube is perfectly elastic, it is generally concluded that, for sufficiently long wavelength disturbances, the entire effect of the tube upon the fluid can be described by a *tube law* relating the local transmural pressure p_t to the local cross-sectional area A and time t (see e.g. Nicholson, Heiser & Olsen 1967; Bird 1964 private communication reported in Rudinger 1966):

$$p_t \equiv P(A,t). \tag{1.1}$$

We have here assumed that the tube is uniform along its axis. As will be demonstrated in § 2, it is sometimes necessary to retain the time dependence in (1.1), even if the physical properties of the tube are time-independent.

A characteristic feature of flows described by this model is that nonlinear waves steepen until the one-dimensional approximation breaks down locally; at which

† Present address: Department of Engineering Sciences and Applied Mathematics Northwestern University, Evanston, Illinois 60901, U.S.A.

point a discontinuity forms between two sections of tube for which the long-wavelength approximation remains valid. The occurrence of these 'discontinuities' or elastic jumps, which are sometimes also referred to as 'shocks', has been pointed out by many authors (e.g. Skalak 1966). Beam (1968) has interpreted observations of Erlanger and Bramwell (Bramwell 1940) as experimental evidence of the formation of travelling elastic jumps. Stationary jumps have been reported by Griffiths (1971, 1975), Shapiro (1977*a*), Elliott & Dawson (1979) and Kececioglu *et al.* (1981).

Various authors have proposed equations to govern these elastic jumps (e.g. Lambert 1958; Beam 1968), but as Kamm & Shapiro (1979) have pointed out, there is no fully satisfactory theory either for the structure of elastic jumps or for the change in state across such jumps. The aim of this paper is to tackle the latter problem by accounting satisfactorily for the elastic balance within the jump.

This elastic balance will naturally depend on the tethering forces applied to the surface of the tube. Further, the tube law (1.1) is an important part of the one-dimensional analysis from which an elastic jump can arise and will be assumed to hold far up- and far downstream of an elastic jump. Hence in § 2 we first verify that for two types of tethering a tube law can be recovered for quasi-one-dimensional axisymmetric disturbances. Then in § 3, an integral relation is derived between the tube law and the longitudinal component of the force exerted by the transmural pressure on a tube of non-uniform cross-section. The proofs given remain valid for collapsed tubes on the assumption that the tube laws exist for sufficiently quasi-one-dimensional disturbances. Further they are also valid even when the tube law does *not* describe the elastic balance at all points of the tube.

The results of § 3 are combined with mass and momentum conservation in § 4, in order to find conditions relating the far up- and far downstream flows of turbulent elastic jumps. Finally in § 5, a discussion of the mechanism of energy loss is given.

2. Tube laws

We shall assume that all deformations are adiabatic, and that the tube walls are perfectly elastic, homogeneous and membraneous. In the case of unsteady deformations we will assume that the inertia of the wall is sufficiently small in order that the elasticity can be based on a quasi-steady analysis. For example, a scaling argument demonstrates that a sufficient condition for this assumption to be valid for an elastic jump propagating into still fluid is

$$\frac{\rho_m h_0}{\rho a_0} \ll 1, \tag{2.1}$$

where ρ is the density of the fluid, and ρ_m, h_0 and a_0 are respectively the density, thickness and radius of the undeformed tube wall. This condition is normally satisfied by human or canine arteries (see e.g. Caro, Pedley & Seed 1974; McDonald 1974).

We will consider two types of tethered tubes:

(i) tubes that are untethered except at their ends;

(ii) tubes that are maintained at a predetermined uniform axial strain during any deformation, by an axial tethering force which is capable of preventing axial movement from the uniform stretched position (cf. Hawley 1970). For ease of reference we will henceforth refer to such constrained tubes as 'longitudinally tethered' tubes.

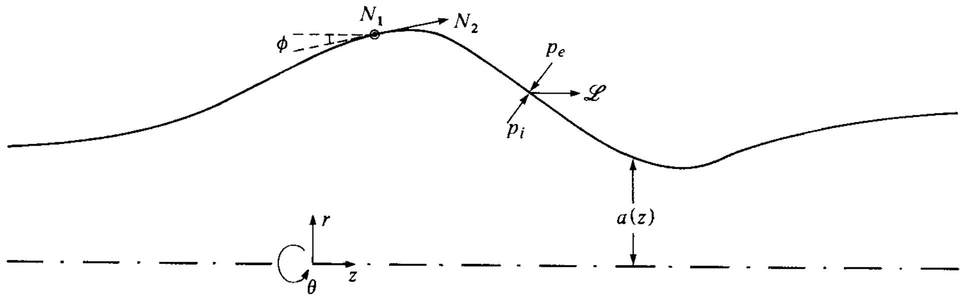

FIGURE 1. Diagram of the forces acting on the membrane. ——, deformed position of membrane; – · –, axis of revolution.

Because physiological vessels, such as arteries, experience considerable longitudinal constraint *in situ* (McDonald 1974), the latter form of tethering is expected to model such vessels more accurately than the former.

In this section we will concentrate on *axisymmetric* deformations of a circular tube, as these lead to simple elastic-balance equations from which the tube laws for both longitudinally tethered and untethered tubes can be derived. However, on the basis of these results, we will postulate the form of the tube laws in the case of sufficiently long wavelength non-axisymmetric deformations.

Axisymmetric deformation of a circular elastic cylinder

We choose cylindrical polar co-ordinates (r, θ, z) as illustrated in figure 1. We denote by $p_i(z)$ and $p_e(z)$ the internal and external pressures which generate the deformation, and by $a(z)$ the deformed radius of the membrane. We also define $N_1(z)$ and $N_2(z)$ as the azimuthal and longitudinal stress resultants (see figure 1), and $\mathscr{L}(z)$ as the imposed axial load per unit area acting over the tube's surface (i.e. the possible tethering force). Then the static balance equations are (see e.g. Flügge 1973)

$$\sin\phi\, N_1 - \cos\phi \frac{d}{dz}(aN_2) = a\cos\phi\, \mathscr{L}, \qquad (2.2a)$$

$$\frac{\cos\phi}{a} N_1 - \cos^3\phi \frac{d^2a}{dz^2} N_2 = p_i - p_e - \sin\phi\, \mathscr{L}, \qquad (2.2b)$$

where
$$\tan\phi = \frac{da}{dz}. \qquad (2.2c)$$

In order to complete the specification, a constitutive relation is required between the stresses and stretches generated by the deformation If the membrane is homogeneous and isotropic then it can be described by a two dimensional work function $W(\lambda_1, \lambda_2)$,† where λ_1 and λ_2 are the principal stretches. Because of the symmetry associated with axisymmetric deformations, λ_1 and λ_2 will be given by stretches of material elements in the azimuthal and longitudinal directions. Consequently, from the general theory of deformations (see e.g. Green & Adkins 1960)

$$N_\alpha = \frac{\lambda_\alpha}{\lambda_1\lambda_2}\frac{\partial W}{\partial\lambda_\alpha}(\lambda_1, \lambda_2) \quad (\alpha = 1, 2). \qquad (2.3)$$

† I am grateful to Dr J. M. Rallison for suggesting the introduction of a two-dimensional work function.

In fact this relation holds if the membrane is not fully isotropic, but displays symmetry of material response only about the planes perpendicular to the θ- and z-axes in its undeformed state (cf. physiological vessels, Patel & Vaishnav 1972). W is then no longer required to be a symmetric function of λ_1 and λ_2, as it is when the membrane is isotropic.

Although we do not non-dimensionalize the variables, we introduce a scaled axial co-ordinate

$$Z = \delta z, \tag{2.4a}$$

where δ is the ratio of a typical radius to a typical axial length scale. For quasi-one-dimensional disturbances

$$\delta \ll 1. \tag{2.4b}$$

Using (2.4*b*) it is possible to deduce tube laws for both longitudinally tethered and untethered tubes.

(*a*) *Longitudinally tethered tubes.* If the imposed uniform axial strain is e, then the principal stretches are given by

$$\lambda_1 = \frac{a}{a_0}, \quad \lambda_2 = \frac{1+e}{\cos\phi}. \tag{2.5}$$

Substituting (2.3)–(2.5) into (2.2) and expanding in powers of δ, we obtain

$$p_i - p_e = \frac{1}{a(1+e)}\frac{\partial W}{\partial \lambda_1}(\lambda_1, 1+e) + O(\delta^2) \tag{2.6a}$$

$$= P_T(A, e) + O(\delta^2), \tag{2.6b}$$

where

$$A = \pi a^2. \tag{2.7}$$

Thus a tube law is recovered in the form of a pressure/area relation independent of time. Further, from (2.2*a*), (2.3), (2.4), $\mathscr{L} = O(\delta)$ and is thus small, although important.

Equation (2.6*a*) represents a balance between the hoop stresses and the transmural pressure (Nicholson, Heiser & Olsen 1967), and is the same relation between pressure and area as is found for a *uniformly* pressurized (untethered) circular elastic tube of fixed length. This is not surprising as a material element of both a longitudinally tethered tube and a uniformly pressurized tube has the same axial length to $O(\delta^2)$. Further, the axial length of a material element remains approximately constant for slowly varying deformations if the tube is collapsed. We might therefore also expect a tube law to hold, and to take the form of a pressure/area relation, for longitudinally tethered collapsed tubes. We note, however, that in this case the elastic balance will include important contributions from both the hoop stresses and the circumferential bending moments. Further, these forces will in general be much smaller than the longitudinal tension. Hence, in order that the longitudinal tension does not contribute significantly to the balance of transmural pressure, (2.4*b*) will have to be strengthened, for example, at least to $\delta \ll h_0/a_0$ if $e \neq 0$.

(*b*) *Untethered tubes.* In this case

$$\mathscr{L} = 0, \quad \lambda_1 = \frac{a}{a_0}. \tag{2.8}$$

Using (2.2*c*)–(2.3) and (2.8), a first integral to (2.2*a*), which is also a first integral of the energy equation, can be found, viz

$$\lambda_2 \frac{\partial W}{\partial \lambda_2} - W = f(t). \tag{2.9}$$

The time dependence of f is associated with any unsteadiness of the deformations (which nevertheless vary sufficiently slowly with time to justify a quasisteady elastic balance).

The value f takes will, in general, depend both on the deformation over the entire length of the tube and on the conditions imposed on the ends of the tube. For example, if the tube is of undeformed length L_0 and is attached to rigid ends a fixed distance L apart, f is determined by (i) the areas imposed at the ends, (ii) the longitudinal pressure distribution, and (iii) a condition expressing conservation of mass, *viz*

$$\int_0^L \frac{1}{\lambda_2 \cos\phi} dz = L_0. \tag{2.10}$$

In order to deduce a tube law, we consider the limit $\delta \to 0$. Then expanding (2.2*b*) in powers of δ, we have using (2.2*c*)–(2.4), that

$$p_i - p_e = \frac{1}{a_0 \lambda_1 \lambda_2} \frac{\partial W}{\partial \lambda_1}(\lambda_1, \lambda_2) + O(\delta^2). \tag{2.11a}$$

From (2.7)–(2.9), this can be expressed in the form

$$p_i - p_e = P_U(A, f) + O(\delta^2). \tag{2.11b}$$

A time-dependent tube law is thus recovered, even though the physical properties of the tube do not vary in time. In a particular steady state or if f is constant, (2.11) reduces to a pressure/area relation, although one that is not the same as (2.6). This difference arises because, whereas for longitudinally tethered tubes the imposed axial load $\mathscr{L}$ maintains the tube at constant axial strain, for untethered tubes a change in axial strain with changing radius is necessary to ensure the longitudinal elastic balance.

In the case of untethered collapsed tubes, the leading-order cross-sectional balance for deformations that vary sufficiently slowly in the axial direction is expected to be between the transmural pressure, circumferential bending moments and hoop stresses (as for longitudinally tethered collapsed tubes). Under such circumstances we anticipate that a tube law will exist. However, (2.11) alerts us to the fact that there is no *a priori* reason why the tube law should take the form of a pressure/area relation for unsteady deformations, such as those observed in the experiments of Conrad (1969). Further, the difference between (2.6) and (2.11) suggests that even in the case of *steady* experiments on spatially slowly varying untethered collapsed tubes, it may be better to determine the tube law directly from experiment rather than to infer it from that for a uniformly pressurized tube.

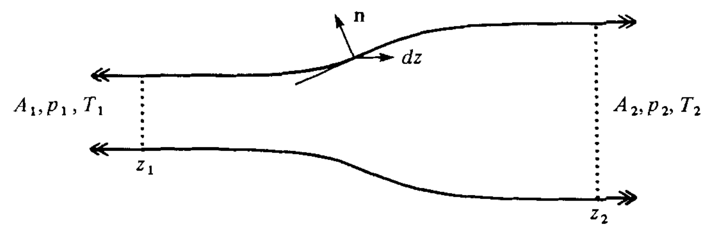

FIGURE 2. Illustration of a deformed elastic tube. The tube is approximately cylindrical for both $z \leqslant z_1$ and $z \geqslant z_2$. T, ⟶⟫, longitudinal tension; A, area; p, uniform pressure.

3. An integral relation

In this section we derive integral expressions relating the tube laws† to the axial component of the force exerted by the transmural pressure on the walls, for both longitudinally tethered and untethered orthotropic elastic tubes. We will assume that as $z \rightarrow \pm\infty$ the transmural pressure becomes approximately constant over the cross-section and sufficiently slowly varying for the tube law to hold (as is the case far up- and downstream of an elastic jump). However for $z = O(1)$, the pressure distribution is allowed to vary sufficiently rapidly in the axial direction that the tube law need not be satisfied at all points.

(a) *Longitudinally tethered tubes*

We begin by supposing that the transmural pressure distribution is uniformly translated to the right by dz. Since the tube is assumed to be longitudinally homogeneous, the deformation similarly translates to the right, although the longitudinal tethering constrains all motion of the tube material elements to be transverse. Further, because the elements of the wall do not move in the direction that the tethering forces $\mathscr{L}$ are applied, these forces do no work. The work done on the membrane between two arbitrary points z_1 and z_2 can therefore be calculated as

$$\int_{\partial S} (p_{\mathrm{i}} - p_{\mathrm{e}})\, \mathbf{n} \,.\, \mathbf{dz}\, dS, \tag{3.1}$$

where $\mathbf{dz}$ is the (axial displacement), $\mathbf{n}$ is the unit normal and ∂S is the surface of the tube between z_1 and z_2 (see figure 2).

However, the material is elastic, and so by definition no matter how it is deformed the work done on it depends only on the initial and final states, and not on the path of deformation. Thus if z_1 and z_2 are chosen where the walls are approximately cylindrical, the work done is equivalent to changing a tube of length dz from area A_2 to area A_1. By supposing this transition were achieved by changing the transmural pressure difference uniformly along the length dz, we see that an alternative expression for the work done is given by

$$-dz \int_{A_1}^{A_2} P_{\mathrm{T}}(A, e)\, dA, \tag{3.2}$$

† As the proofs do not depend on the deformations being axisymmetric, the integral relations will hold for collapsed tubes on the assumption that the elastic balance is described by a tube law for deformations that vary sufficiently slowly in the axial direction.

where $P_T(A, e)$ is the tube law. Equating (3.1) and (3.2) we obtain

$$\int_{\partial S} (p_i - p_e)\,\mathbf{n}\,.\,\mathbf{z}^*\,dS = -\int_{A_1}^{A_2} P_T(A, e)\,dA, \tag{3.3}$$

where $\mathbf{z}^*$ is the unit axial vector. This relation is valid whether or not the tube law holds at all points between z_1 and z_2.

(*b*) *Untethered tubes*†

First we define T_1 and T_2 to be the total longitudinal forces exerted over the cross-sections at z_1 and z_2 respectively – see figure 2. Then because the deformations are assumed to be sufficiently slowly varying at z_1 and z_2 for a tube law to hold to leading order, it also follows that, to leading order, the longitudinal force T_α ($\alpha = 1, 2$) is a function of area A_α and transmural pressure $p_{t\alpha}$, i.e.

$$T_\alpha \equiv T_\alpha(A_\alpha, p_{t\alpha}) \quad (\alpha = 1, 2). \tag{3.4}$$

From the horizontal-force balance we therefore have, whatever the pressure distribution for $z = O(1)$,

$$\int_{\partial S} (p_i - p_e)\,\mathbf{n}\,.\,\mathbf{z}^*\,dS = T_1 - T_2 \equiv g(A_1, A_2, p_{t1}, p_{t2}), \tag{3.5}$$

for some function g which is to be determined.

At those points where the tube law is valid

$$p_i - p_e = P_U(A, f), \tag{3.6}$$

where f is some parameter (for example it could be the longitudinal stretch at some specified area, or alternatively for axisymmetric deformations it might be defined as in (2.9)). In the case of axisymmetric deformations we can show (from (2.9)) that the tube law is satisfied for the same value of f at both z_1 and z_2; in the case of collapsed tubes we only consider cases for which this is so.‡ Then we can evaluate g by considering the *particular example* in which the pressure distribution is sufficiently slowly varying for the tube law to be satisfied at all points between z_1 and z_2. We deduce that (cf. (3.3))

$$\int_{\partial S} (p_i - p_e)\,\mathbf{n}\,.\,\mathbf{z}^*\,dS = -\int_{A_1}^{A_2} P_U(A, f)\,dA. \tag{3.7}$$

Subject to the assumptions made for collapsed tubes, this relation is valid whether or not the tube law is satisfied at all points along the tube.

The proofs leading to (3.3) and (3.7) in fact generalize for the case of axially uniform, thick-walled tubes§ that are in quasi-static equilibrium, if either (*a*) the tubes are tethered by axial surface loads so as to prevent axial motion of the tube at the points

† We reiterate that by an 'untethered' tube, we mean a tube on which the only external load, other than the axial forces at the far ends of the tube, is a pressure force.

‡ We believe that it may be possible to demonstrate, using workfunctions and the elastic balance equations, that even for *collapsed* membraneous elastic tubes the tube law is always satisfied for the same values of f at both z_1 and z_2 (in previous work, where the possible variation of the tube law with f has been overlooked, this question has not arisen).

§ We include in the definition of thick-walled tubes the case of a block of elastic material through which a tube has been bored; such tubes have been used by Kamm (private communication) in recent experiments.

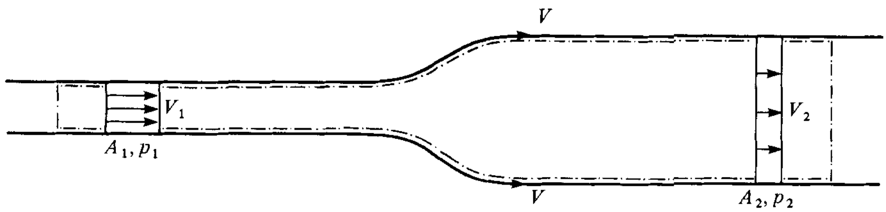

FIGURE 3. Diagrammatic illustration of an elastic jump. In this frame the jump is stationary, and the tube walls move with horizontal velocity V. The long-wavelength limit is valid far up- and far downstream. – · –, control surface.

where the loads are applied (i.e. the generalization of longitudinal tethering), or if (b) the tubes are untethered. The left hand sides of (3.3) and (3.7) need, however, to be replaced by

$$\int_{\partial S_\mathrm{i}} p_\mathrm{i}\,\mathbf{n}\,.\,\mathbf{z}^*\,dS - \int_{\partial S_\mathrm{e}} p_\mathrm{e}\,\mathbf{n}\,.\,\mathbf{z}^*\,dS, \tag{3.8}$$

where ∂S_i and ∂S_e are respectively the inner and outer surfaces of the deformed tube wall.

For the case of a compressible material, it is also necessary to assume that the external pressure is uniform. The proofs then remain straightforward, although the tube laws need to be modified so that they become functions of the external pressure in addition to area, etc. (see e.g. Love 1944).

4. Elastic jumps

The concept of an elastic jump is analogous to that of a hydraulic jump in shallow water theory and a shock wave in gas flow down uniform rigid tubes. It is therefore a transition region, where the quasi-one-dimensional assumption has broken down, and which lies between two regions where this assumption is still valid.

In order to describe such elastic jumps we follow Beam (1968) and Oates (1975*a*), and consider an elastic jump that is moving steadily with constant shape (cf. also Rayleigh's (1914) treatment of hydraulic jumps). We then choose a reference frame fixed in the jump. The use of 'upstream' and 'downstream' with reference to the jump will be in the context of this frame. A control surface is also chosen just inside the tube wall, as illustrated in figure 3.

We assume that the fluid is incompressible, and that the flow is irrotational and approximately inviscid.† Writing A_1, V_1 and p_1 for the upstream area, velocity and internal pressure respectively, and A_2, V_2 and p_2 for the corresponding values far downstream, we then have from conservation of mass and momentum

$$A_1V_1 = A_2V_2 = Q, \tag{4.1}$$

$$\rho A_1V_1^2 - \rho A_2V_2^2 = p_2A_2 - p_1A_1 + \int_{\partial S} p_\mathrm{i}\,\mathbf{n}\,.\,\mathbf{z}^*\,dS. \tag{4.2}$$

† Although viscous stresses on the walls and all viscous dissipation far up- and downstream will be neglected, viscous dissipation associated with turbulence in the vicinity of the jump will be important.

As in §§ 2–3, we restrict attention to longitudinally tethered or untethered elastic tubes. Then from the quasi-one-dimensional assumption far up- and far downstream,

$$p_1 - p_e = P(A_1), \quad p_2 - p_e = P(A_2). \tag{4.3}$$

In the case of untethered tubes, any possible time dependence in the tube law has been suppressed on the assumption that the time scale is long compared with the time it takes for fluid to pass through the jump† (in fact if this were not the case the assumption that the elastic jump were approximately steady would be invalid). Using (3.3) or (3.7) the integral in (4.2) can be simplified. So if the external pressure is constant, substituting (4.1), (4.3) and (3.3) or (3.7) into (4.2), and integrating by parts, we obtain

$$\Phi(A_1) = \Phi(A_2), \tag{4.4a}$$

where

$$\Phi(A) = \frac{Q^2}{A} + \int^A \frac{A}{\rho} \frac{dP}{dA} dA. \tag{4.4b}$$

This is the same result as that of Beam (1968) and Oates (1975*a*); however, because of the arguments presented in § 3 it has been necessary to assume *neither* that the tube is axisymmetric *nor* that the tube law holds within the jump. Equation (4.4) can also be recovered by applying the 'weak-solution method' (Lax 1954) to the governing equations of motion for long-wavelength flows, when these equations have been written in the correct form to express both conservation of mass and momentum flux.

Elliott & Dawson (1979) have made experimental measurements of elastic jumps generated on a (collapsed) elastic tube lying on a flat-bottomed sink (and thus at least partially tethered). Using a pressure/area relation measured in a separate experiment in which the transmural pressure was uniform along the tube, they showed that their results were in broad agreement with (4.4), although they also included a small empirical friction term to improve accuracy.

It is convenient at this point to introduce the speed index F of uniform flows with a volume flux Q (see e.g. Shapiro 1977*a*):

$$F(A) = \frac{\text{local fluid velocity}}{\text{local wave velocity}} = Q\left(\frac{A^3}{\rho}\frac{dP}{dA}\right)^{-\frac{1}{2}}. \tag{4.5}$$

Then, as is conventional, the flow is said to be supercritical or subcritical as F is greater or less than one.

By differentiating (4.4*b*) and using (4.5), Oates (1975*a*) has previously shown that if there is to be a non-trivial solution to (4.4*a*), there must exist A_Q lying between A_1 and A_2 such that

$$\frac{d\Phi}{dA}(A_Q) = 0 \quad (\Rightarrow F(A_Q) = 1). \tag{4.6}$$

Furthermore, Oates has demonstrated that the energy change per unit time across the jump can be written as

$$\text{energy change per unit time} = -Q\int_{A_1}^{A_2}\left(\frac{1}{A} - \frac{1}{A_Q}\right)\frac{d\Phi}{dA} dA. \tag{4.7}$$

† By means of a scaling argument, this can be shown to be so in the case of jumps that form as a result of the steepening of long nonlinear waves.

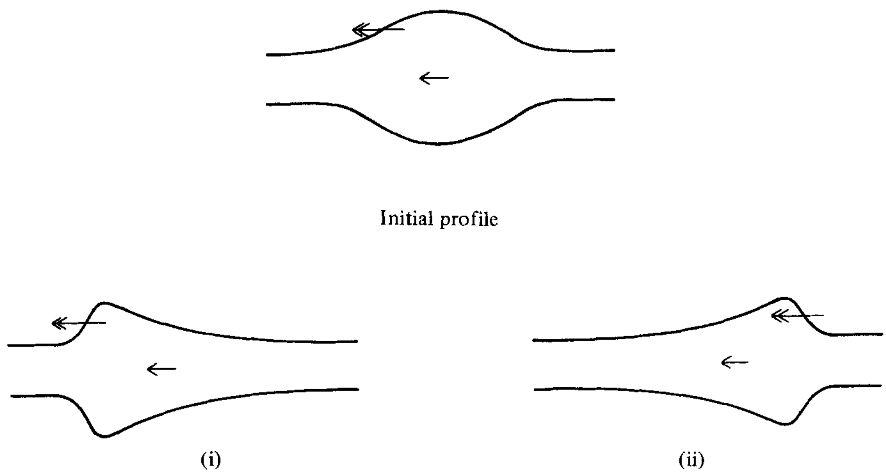

FIGURE 4. The steepening of a simple wave to form an elastic jump. The direction of steepening depends on the elastic properties of the tube. (i) $D^2P > 0$, the jump forms on the front of the wave. (ii) $D^2P < 0$, the jump forms on the rear of the wave. $\twoheadleftarrow$, wave velocity; $\leftarrow$, fluid velocity.

Because the system is causal and the viscous forces on the tube wall are assumed to do a negligible amount of work, energy must be dissipated in the jump. With this condition, and assuming A_Q to be unique, we see from (4.7) that

$$\left.\begin{array}{lll} \text{either} & A_1 < A_2 \quad \text{and} & A_Q \text{ is a minimum of } \Phi, \\ \text{or} & A_1 > A_2 \quad \text{and} & A_Q \text{ is a maximum of } \Phi. \end{array}\right\} \tag{4.8}$$

Oates (1975*a*) did not allow for the latter possibility. Nevertheless, his conclusion that the flow is always supercritical far upstream and subcritical far downstream can still be shown to hold from (4.4*b*–4.6).

From (4.4*b*) and (4.6), we also see that A_Q is a minimum or maximum of Φ according as

$$\frac{d^2\Phi}{dA^2}(A_Q) = \frac{1}{\rho A_Q^2} D^2P(A_Q) \lessgtr 0, \tag{4.9a}$$

where

$$D^2 = \frac{d}{dA} A^3 \frac{d}{dA}. \tag{4.9b}$$

A sufficient condition for A_Q to be unique is thus that D^2P is single-signed. Further, in those circumstances when a tube law reduces to a pressure/area relation, Nicholson *et al.* (1967) have deduced that compression waves steepen and rarefaction waves flatten, or vice versa, according as

$$D^2P \gtrless 0. \tag{4.10}$$

Equations (4.8)–(4.10) are hence seen to be consistent; for if $D^2P > 0$, compression waves steepen and an elastic jump is required with $A_1 < A_2$, while if $D^2P < 0$, rare faction waves steepen and a jump is required with $A_1 > A_2$ – see figure 4. Although Oates (1975*a*, *b*) came near to the above result, he assumed that the speed index always decreases with increasing area; whereas from (4.5),

$$\frac{dF^2}{dA} = -\frac{\rho Q^2 D^2 P}{(A^3 dP/dA)^2}. \tag{4.11}$$

Thus this is only true if $D^2P > 0$. When $D^2P < 0$, the reverse holds, and so for the same flow rate a supercritical flow has a larger cross sectional area than a subcritical one.†

If A_Q were not unique, D^2P could not be single-signed, and the final solution would be the result of an interaction of the above opposing effects.

5. The energy loss

The proposed solution for elastic jumps will only be valid for jumps in which the necessary energy loss can be accounted for. Moreover, in performing the stress balance in the tube wall we have assumed that the wall remains perfectly elastic and that the tethering mechanism does not break down. No energy loss can therefore take place in the wall. We hence expect that the major cause of energy loss for elastic jumps described by the above theory will be turbulent dissipation.

Furthermore, although it is consistent to ignore the effects of longitudinally moving boundaries in the calculations of the mass and momentum fluxes for a given turbulent jump, the actual existence of a turbulent jump may depend on the velocity of the walls. As illustrations we will consider the particular examples in which (1) the jump and walls are both at rest and (2) the upstream wall and fluid velocities are equal (as for a jump propagating into fluid at rest).

Jump and walls at rest

This case naturally subdivides according as $A_1 \lessgtr A_2$.

(*a*) $A_1 < A_2$. The adverse pressure gradient present within the enlargement is expected to lead, for all but the weakest of jumps, to flow reversal within the wall boundary layer, separation, and for sufficiently large Reynolds numbers the generation of a turbulent region; in that case the above theory is expected to yield an accurate description. The separated region will however be different from that which occurs at the *rapid* enlargement of a rigid tube. For, in the case of a rigid tube, the pressure on the tube wall is assumed approximately constant as the result of the formation of a 'steady' jet and a surrounding slow-moving region at the expansion. Consequently the Borda–Carnot 'shock' condition holds (see for example Batchelor 1967). This condition predicts a smaller downstream pressure recovery than (4.4). It apparently cannot hold for an elastic tube because the elastic-balance equations cannot be satisfied if there is a constant transmural pressure distribution at the expansion.

A possible explanation for the departure from the Borda–Carnot condition is that the separation at the expansion is intermittent, with separated regions forming and then disappearing as vortices are shed (cf. the 'large transitionary stall regime' described by Reneau, Johnston & Kline (1967) for a two-dimensional diffuser). Further,

† The theory of long-wave fluid flow down a channel of arbitrary cross-section is analogous to much of the work of this section. That problem is characterized by a function $h(A)$ relating the surface elevation h to the cross-sectional area A it encloses. $h(A)$ plays an equivalent role to the tube law. For example, long waves steepen at the front/back, and for a specified flow rate the Froude number decreases/increases with elevation, as $D^2h \gtrless 0$, cf. (4.10)–(4.11). Furthermore, for a hydraulic jump to be dissipative, the upstream area A_1 and the downstream area A_2 must satisfy, $A_2/A_1 \gtrless 1$ according to the same criterion.

Kamm (1981, private communication) has observed relatively large amplitude oscillations at the expansion for certain flow rates. These oscillations may be the result of a resonant interaction between the wall's own elastic vibrations and the forcing due to the pressure fluctuations associated with intermittent separation.

(*b*) $A_1 > A_2$. In this case there is a favourable pressure gradient associated with the mean flow, and turbulence downstream of the contraction is unlikely. Upstream separation may occur (see Smith 1977, 1978), but the contraction then has to be relatively more severe compared with the expansion needed to generate downstream separation in the previous case.† When upstream separation is present, an instability of the free shear layer may lead to the turbulence necessary for the above theory to be valid.

Upstream wall and fluid velocities equal

Such jumps will eventually be generated by the steepening of a pulse propagating along a tube of otherwise undisturbed fluid, assuming viscous effects remain negligible until the jump is formed. No boundary-layer separation is envisaged. For example, consider the case $A_1 < A_2$. Then the longitudinal velocity of the walls will everywhere be greater than or equal to that of the fluid (at least for a longitudinally tethered tube). Consequently, flow reversal in the boundary layer will not occur without flow reversal in the core.

If turbulent elastic jumps exist, we therefore expect the generation of the turbulence to be associated with an instability of the inviscid flow. However, because of the presence of a material boundary in place of a free surface, there is no direct parallel to the breaking waves that lead to turbulence in the analogous turbulent bores.

A further mechanism of energy loss for elastic jumps, which has been suggested by Pedley (1980), is radiation through a (stationary) wave train. Wavetrains have been observed upstream of steady turbulent elastic jumps by Kececioglu *et al.* (1981). The wavetrains associated with turbulent elastic jumps on axisymmetrically deformed tubes will be studied in a subsequent paper (see also Cowley (1981) and McClurken *et al.* (1981)). Here we note that modifications will be necessary to (4.4) if a wavetrain exists, as the flow will no longer be uniform both far up- and far downstream.

Equation (4.4) will also not be accurate for jumps within which there is no turbulence. Two alternatives are then possible. The first is that either viscous or unsteadiness effects have become important, and that a 'laminar' elastic jump has formed. This possibility has been examined by Cowley (1981) for the case of jumps propagating into fluid at rest. The second possibility is that the elasticity and/or tethering assumptions have broken down. Examples of this case are the viscoelastic jumps studied by Kivity & Collins (1974), Johnson (1971) and Buggisch (1980).

6. Conclusions

By comparing the tube laws for both longitudinally tethered and untethered axisymmetric elastic tubes, we have illustrated that the form of the tube law depends (crucially) on the nature of the tethering. For, in the case of a longitudinally tethered tube, the tube law was shown to be a function only of area and axial prestrain, while

† This is illustrated in the design of a Venturi meter, for which the contraction is in general more severe than the expansion.

for an untethered tube it was found in general to be a function of both area and time. Furthermore, in those circumstances under which the tube law for an untethered tube is time-independent, it still in general has a different form to that for a longitudinally tethered tube. We therefore suggest that the use of pressure/area relations, derived under conditions of uniform axial prestrain, to describe the (unsteady) behaviour of an untethered tube should be re-examined (see e.g. Rudinger 1970; Kamm & Shapiro 1979).

Without neglecting the effects of longitudinal tension, we have also derived a condition relating the volume flux to the up- and downstream areas for a turbulent elastic jump. Causality has then been invoked in order to determine whether the ratio of up- to downstream areas is greater or less than one. The result so obtained is in agreement with that required for causal elastic jumps to form as nonlinear waves steepen.

A preliminary discussion of the effects of the velocity of the walls relative to the jump has also been given. For while the wall velocity has little effect on the jump conditions for a fluid of small viscosity, it may play a role in the actual existence of such jumps. For example, separation and a jump may be more likely if the walls are at rest than if they are moving downstream relative to the jump.

The derived jump conditions are believed to be appropriate descriptions of the elastic jumps observed in the experiments of Griffiths (1971, 1975), Elliott & Dawson (1979) and Kamm & Shapiro (1979). We note, however, that we have taken no account of the viscous forces on the wall. As Elliott & Dawson (1979) have observed, in the case of stationary jumps these forces will lead to smaller pressure recoveries than those predicted by (4.4). Wall friction is also thought to explain partially the difference between (4.4) and the somewhat scattered experimental measurements of Kececioglu *et al.* (1981). A further important reason for the disparity between these experiments and (4.4) is the presence of wavetrains upstream of the jumps. The existence of these wavetrains means that there is no position upstream of the jump at which the tube is approximately uniform, and which is reasonably close to the jump. We also note that another (minor) reason for the disparity may be that, whereas the experiments were performed on 'untethered' tubes, the tube law used in (4.4) was that for a uniformly inflated tube (see § 2).

Equation (4.4) is also expected to be appropriate to the jump like transitions which occur in the urethra (Griffiths 1980) and the airways from the lung during forced expiration (Dawson & Elliott 1977). Further, numerical integration of the long-wavelength nonlinear model by Anliker, Rockwell & Ogden (1971), using experimental data for the boundary conditions, has indicated the probable formation of jumps in the arterial system if there is aortic insufficiency.† This result is supported by the work of Rudinger (1970), Bryant-Moodie & Haddow (1977) and Pedley (1980), who have studied the propagation of simple waves along an aorta modelled as a uniform elastic tube. They predicted the formation of jumps between 43 and 100 cm, from the heart normally, but within 20 cm for a diseased system. However, because these calculations ignore the branching and changing properties of the arterial system, the distances are probably an underestimate (Sugawara *et al.* 1973). Hence in practice

† Aortic insufficiency, i.e. incompetent or leaking aortic valves, results in the extensive regurgitation of arterial blood from the aorta back into the left ventricle and the consequent generation of extremely large pressure pulses by the heart in order to maintain the necessary cardiac output per beat.

jumps are expected to form only in diseased systems. Such jumps are thought to provide an explanation of the so-called 'pistol-shot' sounds heard in patients with incompetent heart valves. If these arterial jumps are sufficiently strong to result in turbulent flow, a synthesis of our work with that on viscoelastic jumps by Kivity & Collins (1974) is expected to provide the best description.

I would like to thank Dr T. J. Pedley for his guidance and encouragement; Professor Sir James Lighthill for a useful discussion on § 4; and Professor A. Shapiro, Dr R. Kamm and the referees for their comments and suggestions. This work was partially supported by an award from the Science Research Council.

REFERENCES

ANLIKER, M., ROCKWELL, R. L. & OGDEN, E. 1971 Non-linear analysis of flow pulses and shock waves in arteries. Part I and Part II. *Z. angew. Math. Phys.* **22**, 217–246, 563–581.

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BEAM, R. M. 1968 Finite amplitude waves in fluid-filled elastic tubes: wave distortion, shock waves and Korotkoff sounds. *N.A.S.A. Tech. Note* D-480.

BRAMWELL, J. C. 1940 Blood pressure and its estimation. *Lancet* **1**, 138–140, 184–188.

BRYANT-MOODIE, T. & HADDOW, J. B. 1977 Waves in thin-walled elastic tubes containing an incompressible inviscid fluid. *Int. J. Non-linear Mech.* **12**, 223–231.

BUGGISCH, H. 1980 Nonlinear waves. In *Proc. 15th Int. Cong. Theor. Appl. Mech., Preprints, pp.* 57–68. North-Holland.

CARO, C. G., PEDLEY, T. J. & SEED, W. A. 1974 Mechanics of the circulation. In *Cardiovascular Physiology* (ed. A. C. Guyton & C. E. Jones), pp. 1–47. Medical & Technical Publ.

CONRAD, W. A. 1969 Pressure–flow relationships in collapsible tubes. *I.E.E.E. Trans., Bio-Med. Engng* BME **16**, 284–295.

COWLEY, S. J. 1981 High Reynolds number flows through distorted channels and flexible tubes. Ph.D. dissertation, University of Cambridge.

DAWSON, S. V. & ELLIOTT, E. A. 1977 Wave speed limitation on expiratory flow – a unifying concept. *J. Appl. Physiol., Respirat. Environ. Exercise Physiol.* **43**, 498–515.

ELLIOTT, E. A. & DAWSON, S. V. 1979 Fluid velocity greater than wavespeed and transition from supercritical to subcritical flow in elastic tubes. *Med. Biol. Engng Comp.* **17**, 192–198.

FLÜGGE, W. 1973 *Stresses in Shells*. Springer.

GREEN, A. E. & ADKINS, J. E. 1960 *Large Elastic Deformations*. Oxford University Press.

GRIFFITHS, D. J. 1971 Hydrodynamics of male micturition. I. Theory of steady flow through elastic walled tubes. *Med. Biol. Engng* **9**, 581–588.

GRIFFITHS, D. J. 1975 Negative-resistance effects in flow through collapsible tubes. I. Relaxation oscillations. *Med. Biol. Engng* **13**, 785–790.

GRIFFITHS, D. J. 1980 *Urodynamics*. Adam Hilger.

HAWLEY, J. K. 1970 Studies of mathematical models of aortic blood flow. Ph.D. dissertation, Rensselaer Polytechnic Institute, Troy, New York.

JOHNSON, R. S. 1971 Nonlinear waves in fluid filled elastic tubes and related phenomenon. Ph.D. dissertation, University of London.

KAMM, R. D. & SHAPIRO, A. H. 1979 Unsteady flow in a collapsible tube subjected to external pressure or body forces. *J. Fluid Mech.* **95**, 1–78.

KECECIOGLU, I., MCCLURKEN, M. E., KAMM, R. D. & SHAPIRO, A. H. 1981 Steady, supercritical flow in collapsible tubes. Part 1. Experimental observations. *J. Fluid Mech.* **109**, 367–389.

KIVITY, Y. & COLLINS, R. 1974 Steady state fluid flow in viscoelastic tubes. Application to blood flow in human arteries. *Arch. Mech. (Warsaw)* **26**, 921–931.

LAMBERT, J. W. 1958 On the nonlinearities of fluid flow in non-rigid tubes. *J. Franklin Inst.* **266**, 83–102.

LAX, P. D. 1954 Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Maths.* **7**, 159–193.

LOVE, A. E. H. 1944 *A Treatise on the Mathematical Theory of Elasticity*. Dover.

MCCLURKEN, M. E., KECECIOGLU, I., KAMM, R. D. & SHAPIRO, A. H. 1981 Steady, supercritical flow in collapsible tubes. Part 2. Theoretical studies. *J. Fluid Mech.* **109**, 391–415.

MCDONALD, D. A. 1974 *Blood Flow in Arteries*. Edward Arnold.

NICHOLSON, H. W., HEISER, W. H. & OLSEN, J. H. 1967 Wave propagation in liquid-filled elastic tubes. *Bull. Mech. Engng Educ.* **6**, 371–376.

OATES, G. C. 1975*a* Fluid flow in soft walled tubes. I. Steady flow. *Med. Biol. Engng* **13**, 773–778.

OATES, G. C. 1975*b* Fluid flow in soft walled tubes. II. Behaviour of finite waves. *Med. Biol. Engng* **13**, 779–784.

PATEL, D. J. & VAISHNAV, R. N. 1972 The rheology of large blood vessels. In *Cardiovascular Fluid Dynamics* (ed. D. H. Bergel), vol. 2, pp. 1–64. Academic.

PEDLEY, T. J. 1980 *The Fluid Mechanics of Large Blood Vessels*. Cambridge University Press.

RAYLEIGH, LORD 1914 On the theory of long waves and bores. *Proc. R. Soc. Lond.* A **90**, 324–328.

RENEAU, L. R., JOHNSTON, J. P. & KLINE, S. J. 1967 Performance and design of straight, two-dimensional diffusers. *Trans. A.S.M.E.* D, *J. Basic Engng* **89**, 141–148.

RUDINGER, G. 1966 Review of current mathematical methods for the analysis of blood flow. In *Proc. Biomedical Fluid Mechanics Symp., A.S.M.E., New York*. pp. 1–33.

RUDINGER, G. 1970 Shock waves in mathematical models of the aorta. *Trans. A.S.M.E.* E, *J. Appl. Mech.* **37**, 34–37.

SHAPIRO, A. H. 1977*a* Steady flow in collapsible tubes. *Trans. A.S.M.E.*, K, *J. Biomech. Engng* **99**, 126–147.

SHAPIRO, A. H. 1977*b* Physiologic and medical aspects of flow in collapsible tubes. In *Proc. 6th Canadian Cong. Appl. Mech., Vancouver*, pp. 883–906.

SKALAK, R. 1966 Wave propagation in blood flow. In *Biomechanics* (ed. Y. C. Fung). New York: A.S.M.E.

SMITH, F. T. 1977 Upstream interactions in channel flows. *J. Fluid Mech.* **79**, 631–655.

SMITH, F. T. 1978 Flow through symmetrically constructed tubes. *J. Inst. Math. Appl.* **21**, 145–156.

SUGAWARA, M., SAKURAI, Y., IMACHI, K. & FUJIMASA, I. 1973 Non-linear theory of pulse waves in arteries. *Japan. J. Med. Electronics Biol. Engng* **11**.